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Conditions for Universal Reducibility of a Two-Stage Extremization Problem to a One-Stage Problem

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An extremization problem in the general form $\max_{x \in X} f(x)$ is considered. This notation is treated as the description of a parametric family of problems where the extremized function f is the same but the set X is a variable parameter. Such a "mass" treatment of the extremization problem for function f determines implicitly the "choice transformation" $X \to Y$, where Y is the set of solutions, i.e., $Y = \operatorname{Arg} \max_{x \in X} f(x)$, and X goes over a given family $\mathscr{X}$ of admissible sets. Similarly, a two-stage problem of sequential extremization of two functions, φ and ψ, determines the superposition of two related choice transformations. We consider the cases when the function φ and/or ψ may be vectorial and the extremization is understood in the Pareto sense. The very possibility of reducing a two-stage problem to a one-stage problem having the same solution set Y for every admissible $X \in \mathscr{X}$ is studied. Unlike the usual "lexicographical" extremization of two scalar functions, such a reduction is not always possible in the vectorial case. The necessary and sufficient conditions for it are stated. © 1986 Academic Press, Inc.

1. INTRODUCTION AND STATEMENT OF PROBLEM

We shall consider mainly the problems of extremization (for definiteness, maximization) of scalar and vectorial functions on abstract sets, i.e., the problems in the form:

$$\max_{x \in X} f(x). \tag{1}$$

In this way we shall treat the formulation (1) in two ways: (1) as the *individual* problem of extremization of the given function f on the given set X, and (2) as the *mass* problem of extremization of the same function f on various sets X in which role the members of some given family $\mathscr{X}$ of

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admissible sets are taken. In what follows, under the *problem of extremization for function* f (briefly, f-problem), we shall understand the mass problem of extremization for f in the above-mentioned sense. That is, we shall virtually consider the parameterized (by X) family of individual problems in the form (1) which are different in the parameter $X \in \mathscr{X}$. We shall be interested in the behavior of the set Y of solutions of the f-problem (1) under all possible X's, i.e., in the form of transformation $X \to Y$ implicitly defined due to (1) where $X \in \mathscr{X}$ and

$$Y = \operatorname{Arg} \max_{x \in X} f(x). \tag{2}$$

Indeed, similar transformations converting every admissible set X into some subset $Y \subseteq X$ really appear as a result of applying certain "mass" procedures to the sets $X \in \mathscr{X}$ (such procedures may specifically include the use of extremizing operations for various functions). Every problem (procedure) which explicitly or implicitly selects from any set $X \in \mathscr{X}$ some subset Y is called a *choice problem* on $X \in \mathscr{X}$. In these terms the simplest f-problem of extremization (1) may be considered as a paradigm of choice problems. The aim of the present work is to consider some other problems having more complicated forms and to collate them with this paradigm, being interested in the possibility or impossibility of finding the "equivalent" paradigm problem for any such starting problem (the sense of "equivalence" being strictly defined below).

As the starting problems we shall furthermore consider the problems of sequential extremization of two functions, φ and ψ, having the same domain U. We shall assume that under any fixed X ($X \subseteq U$) used as the admissible (feasible) set for the *first stage* of extremization, viz, the φ-problem, the received solution set V for this φ-problem is used further as the admissible set for the *second stage* of extremization, viz, the ψ-problem. The resulting overall problem

$$\max_{v \in V} \psi(v), \qquad \text{where} \quad V = \operatorname{Arg} \max_{x \in X} \varphi(x), \tag{3}$$

will be called *the two-stage problem of sequential extremization of functions* φ *and* ψ (or the *φ, ψ-problem* for brevity). Such a φ, ψ-problem generates the choice transformation $X \to Y$, $X \in \mathscr{X}$, where

$$Y = \operatorname{Arg} \max_{v \in \operatorname{Arg}\max_{x \in X} \varphi(x)} \psi(x). \tag{4}$$

It is evident that this transformation is the superposition of two "single-stage" choice transformations, $X \to V$ and $V \to Y$, generated by the φ-problem and the ψ-problem, respectively.

Let us pose the question: is it possible to replace a two-stage φ, ψ-problem (3) by some equivalent one-stage f-problem? The answer to this question essentially depends on the way of defining the notion of "equivalent substitution." It is natural to require that the "substitute" problem possess exactly the same set of solutions as the initial one. But if we applied this requirement only to the individual problem on a single fixed set X and if we simultaneously permitted arbitrary selection of any criterial function f in the proposed substitute equivalent f-problem, then the answer to our question should turn out to be trivially positive. It would be sufficient, e.g., to take the characteristic function of the set Y as the f function, i.e., let $f(x)=1$ for $x \in Y$ and $f(x)=0$ otherwise. Nevertheless the above question remains nontrivial if we treat our problems as mass problems of choice; then according to such a treatment we shall consider all admissible sets $X \in \mathscr{X}$, and by the "equivalence of problems" we shall mean the coincidence of their respective solution sets, Y's, for every $X \in \mathscr{X}$.

DEFINITION 1. Consider two choice problems on sets X from the same family $\mathscr{X}$. We shall say that one problem is *universally reducible* to another (and conversely) if for every $X \in \mathscr{X}$ the respective solution sets Y_1 and Y_2 of these two problems coincide.

In what follows, speaking of reducibility of one problem to another (specifically, of two-stage to one-stage), we shall mean the universal reducibility in Definition 1 even if the word "universal" is omitted.

Remark. Although, as we have indicated above, for every choice problem on a fixed X a respective f-problem with the same solution set Y can be designed in a trivial way, such a design by itself yields nothing for the answer to the question on the universal reducibility to an f-problem. Indeed, the so designed critical function f will generally depend parametrically on X, i.e., $f=f_X(x)$. The question is if it is possible to "sew" different functions $f_X(x)$ together to get a joint function $f(x)$ on $\bigcup_{X \in \mathscr{X}} X$ independent of X as is required in the form (1) of the mass f-problem. In the choice theory (see, e.g., [1]) for this question in a general case the negative answer is given: definitely not every choice problem is universally reducible to a scalar or even vectorial extremization problem. The results of analysis of possibilities for such reducibility for two-stage φ, ψ-problems are given below (some of them have been published earlier, particularly in [1, 2]).

In the simplest case, when both φ and ψ are scalar functions, the two-stage φ, ψ-problem (3) is a well-known problem of "lexicographic maximization" (see, e.g., [3]). Such a "scalar–scalar" two-stage problem is always reducible to a one-stage extremization problem (1) with some function f being a scalar one also; we shall discuss it below. But if in the role of φ and/or ψ, one may use vectorial functions then the situation becomes more complicated: universal reducibility of a φ, ψ-problem (3) to an f-problem (1) can demand that certain conditions be fulfilled. The setting of such conditions is the main topic of this paper.

Everywhere below we assume for the sake of simplicity that all considered functions (f, φ, etc.) are defined on some finite set U (i.e., f: $U \to R$ or φ: $U \to R^n$, etc.) and the family $\mathscr{X}$ is the collection of all nonempty subsets X of the set U (i.e., $\mathscr{X} = 2^U \setminus \{\varnothing\}$).

2. Vectorial Extremization

Let us start from statements of extremizaton problems under several criterial functions. Assume that n scalar functions $\varphi_1, ..., \varphi_n$ on U are given. Before looking at the "sequential" application of different criterial functions in extremization procedures, let us discuss their "parallel" application. The "parallel" (i.e., simultaneous and "equitable") usage of functions $\varphi_1, ..., \varphi_n$ as extremization criteria naturally leads to the usual consideration of the "vectorial" criterial function $\varphi = (\varphi_1, ..., \varphi_n)$ when the vector maximization problem

$$\max_{x \in X} \varphi(x) \tag{5}$$

is understood as the problem of finding the solution set

$$\begin{aligned} Y &= \operatorname{Arg} \max_{x \in X} \varphi(x) \\ &= \{ y \in X \mid \text{there is no } x \in X \text{ such that } \varphi(x) > \varphi(y) \}. \end{aligned} \tag{6}$$

Here vectorial inequality for φ is defined as an appropriate generalization of scalar inequality which converts itself into usual numerical inequality in the case $n = 1$. This fact implies that the scalar extremization problem is a particular case of the vectorial one. Following the standard definition we shall treat vectorial inequality as a component-wise problem (a *vectorial superiority* relation in the Pareto sense):

$$\varphi(x) > \varphi(y) \Leftrightarrow (\varphi_i(x) > \varphi_i(y),\ i = 1, ..., n). \tag{7}$$

So Y in (6) is the set of Pareto-maximal points for the vectorial function φ.

Remark. The definition of the Paretian relation of strict vectorial superiority $>$ usally admits that some (but not all) respective componentwise inequalities may be unstrict. For further exposition this difference in two varieties of vectorial inequality $>$ is unessential.

Let us pose the question: is the vectorial maximization problem (5), with $n > 1$, reducible to some scalar maximization problem, with $n = 1$? The answer to this question will serve as an illustration for the notion of universal reducibility for mass choice problems.

First, we shall introduce some notations. The failure (logical negation) of the vectorial inequality $>$ we shall call *vectorial nonsuperiority* and denote by $\not>$ (in the scalar case, $n = 1$, the relation $\not>$ is equivalent to $\leqslant$). Let us also introduce for vectors the following relation of *mutual nonsuperiority* denoted by $><$:

$$\varphi(x) >< \varphi(y) \Leftrightarrow (\varphi(x) \not> \varphi(y) \text{ and } \varphi(x) \not< \varphi(y))$$

(in the scalar case $><$ is equivalent to $=$).

Let us call a triple of elements $u, v, w \in U$ a *φ-triad* if

$$\varphi(u) < \varphi(v), \qquad \varphi(u) >< \varphi(w), \qquad \text{and} \qquad \varphi(v) >< \varphi(w). \tag{8}$$

Note that the system of relations (8) is possible even with $n = 2$, e.g., when $\varphi_1(u) < \varphi_1(v) < \varphi_1(w)$ and $\varphi_2(u) > \varphi_2(v) > \varphi_2(w)$.

Let us satisfy ourselves that under the existence of a φ-triad in U the universal reducibility of the φ-problem (5) to an f-problem (1) with some scalar function f is definitely impossible. Really, on setting $X = \{u, v, w\}$, we get for (5) $Y = \{v, w\}$. Hence in the case of the reducibility of (5) to (1), we might have $f(u) < f(v) = f(w)$. On the other hand, on setting $X = \{u, w\}$, we get for (5) $Y = \{u, w\}$. In the case of the reducibility of (5) to (1) this would imply that $f(u) = f(w)$, in contradiction with the preliminary result $f(u) < f(w)$. Therefore the problem (5) is not reducible to (2).

Remark. Really, the converse is also true: the existence of a φ-triad in U is not only sufficient but also necessary for the irreducibility of a problem (5) to (1).

This well-known fact is easily deduced in Section 5 by using some elementary notions of decision theory.[1] In the main part of the text we shall if possible avoid addressing these notions for the sake of autonomy of exposition. For the same purpose in several cases we give independent proofs of statements known (in one form or other) in the theory of choice and decision making [1–4].

In particular, we shall now give an independent "constructive" proof of the fact that if φ-triads in U are absent then the φ-problem (5) is reducible to an f-problem (1) with some scalar function f. It will be useful for us to

[1] In this case the essence of the matter is the nontransitivity of the mutual nonsuperiority relation $><$ which is just equivalent to the existence of a φ-triad in U.

write the requirement of absence of φ-triads in U in the form of the *Quadruple condition*: for every $x, y, r, s \in U$

$$\varphi(x) > \varphi(y), \qquad \varphi(x) \asymp \varphi(r), \qquad \varphi(y) \asymp \varphi(s) \Rightarrow \varphi(r) > \varphi(s).$$

Really, the existence of a φ-triad $u, v, w \in U$ of the form (8) obviously violates the Quadruple condition (it is sufficient to take $x = r = u$, $y = v$, $s = w$). Conversely, let the Quadruple condition be violated, viz, its left part be fulfilled but its right part be violated (i.e., $\varphi(r) \not> \varphi(s)$). Then, as it is easy to see, under any possible relation between $\varphi(x)$ and $\varphi(s)$ in the quadruple x, y, r, s at least one φ-triad (x, y, s or x, s, r) will exist.

Thus let φ-triads be absent in U so that the Quadruple condition is fulfilled. We shall design the desired function f on U. Define for every $u \in U$ the set

$$E(u) = \{v \in U \mid \varphi(u) \asymp \varphi(v)\}$$

and let

$$f(u) = \frac{1}{|E(u)|} \sum_{v \in E(u)} \sum_{i=1}^{n} \varphi_i(v).$$

We will show that the given scalar function f is equivalent to the vectorial function φ in the sense that

$$\varphi(x) > \varphi(y) \Leftrightarrow f(x) > f(y).$$

Note for the beginning that due to the absence of φ-triads in U

$$\varphi(x) \asymp \varphi(y) \Rightarrow E(x) = E(y),$$

hence

$$\varphi(x) \asymp \varphi(y) \Rightarrow f(x) = f(y).$$

On the other hand, the definition of f in light of the Quadruple condition immediately implies

$$\varphi(x) > \varphi(y) \Rightarrow f(x) > f(y).$$

This together with the previous implication just means the equivalence between f and φ which yields universal reducibility of the φ-problem to the f-problem.

DEFINITION 2. A vectorial extremization problem (5) (with $n > 1$) will be called *essentially vectorial* if it is not universally reducible to an f-problem (1) with any scalar function f.

Now the above statements on reducibility of problem (5) to (1) may be summarized in the following Lemma.

LEMMA 1. *For a problem of extremization of a vectorial function φ on U (with $\mathscr{X} = 2^U \setminus \{\varnothing\}$) to be essentially vectorial, it is necessary and sufficient that φ-triads were in U.*

Remark. We once more emphasize that nonreducibility of a given vectorial extremization problem to a scalar one is generated by the requirement of universality of such a reducibility on all $X \in \mathscr{X}$. In contradistinction to it, the statements on the reducibility of a vectorial extremization problem to a scalar one on a single fixed set X are sometimes considered. These statements can be nontrivial if we set some a priori requirements on the form of the scalar function sought (e.g., the representability in the form of linear combination of components of the initial vectorial function).

3. SEQUENTIAL EXTREMIZATION: THE SIMPLE CASES

Now let various criterial functions, scalar or vectorial, be used in extremization procedures sequentially. We shall consider the two-stage procedure in the form of a φ, ψ-problem (3) with criterial functions φ and ψ at the first and the second stages of extremization, respectively. The four types of problems depend on the types of functions φ and ψ:

I. φ scalar, ψ scalar;
II. φ scalar, ψ vectorial;
III. φ vectorial, ψ scalar;
IV. φ vectorial, ψ vectorial.

The logical subordination of these types of problems is shown as:

$$\begin{tikzcd} & \text{I (scalar–scalar)} \arrow[dl] \arrow[dr] & \\ \text{II (scalar–vectorial)} \arrow[dr] & & \text{III (vectorial–scalar)} \arrow[dl] \\ & \text{IV (vectorial–vectorial)} & \end{tikzcd}$$

Here every lower type of problem includes as a particular case every higher type. In this section we shall consider problem types I and II in which reducibility to one-stage extremization may be verified directly. The more complex types III and IV will be considered in the later sections.

I. Let us start with a scalar–scalar problem (type I, "lexicographic scalar extremization") [4]. We shall display its universal reducibility to a scalar extremization problem by using the following self-explanatory construction. Let $|U| = N$. Then without loss of generality we may consider values φ and ψ to be integers between 0 and $N-1$ and index them by N-ary digits. (Here we as usual base our results on the evident fact that functions under extremization admit any monotone transformations without changing the solutions for extremization problems).

Consider now two-digit numbers in the N-ary positional calculus system having the form $\langle \varphi, \psi \rangle$, i.e., where digit (one-digit number) φ is set in a higher position and ψ in a lower one. Let us construct now a new scalar function χ on the set U having values

$$\chi(x) = \langle \varphi(x), \psi(x) \rangle, \qquad x \in U, \tag{9}$$

and consider the extremization problem

$$\max_{x \in X} \chi(x). \tag{10}$$

It is easy to make sure that the φ, ψ-problem (3) with given scalar functions φ and ψ is universally reduced to the χ-problem (10). Indeed, the set $Y = \operatorname{Arg} \max_{x \in X} \chi(x)$ obviously consists of those and only those elements $y \in X$ which satisfy the two conditions:

(1) among elements of X, they have the maximal value of the highest digit for χ, i.e., the maximal value of φ;

(2) among these φ-maximal elements of X, they have the maximal value of the lowest digit for χ, i.e., the maximal value of ψ.

But this is exactly the description of the set Y given by (4).

II. Now, let in a two-stage φ, ψ-problem (3), the function φ be a scalar and ψ be a vectorial one: $\psi = (\psi_1, ..., \psi_n)$. Repeating the process of introducing two-digit N-ary numbers, we shall define a vectorial function $\chi = (\chi_1, ..., \chi_n)$ by the equalities

$$\chi_i(x) = \langle \varphi(x), \psi_i(x) \rangle, \qquad i = 1, ..., n. \tag{11}$$

Again it is possible to verify that a two-stage φ, ψ-problem (3) is reduced to the one-stage χ-problem (10) but now with a vectorial function χ of the form (11). For this purpose we shall consider an arbitrary element $y \in X$ and show that it is not a solution of the problem (10) if and only if it is not a solution of the problem (3).

Denote $\varphi^* = \max_{x \in X} \varphi(x)$. Then the set V defined by (3) is $V = \{v \in X \mid \varphi(v) = \varphi^*\}$. For $y \in X$ not to be a solution of the χ-problem (10), it is necessary and sufficient that for some $z \in X$, the inequality $\chi(y) < \chi(z)$ holds, i.e.,

$$\begin{aligned} &(\langle \varphi(y), \psi_1(y) \rangle, ..., \langle \varphi(y), \psi_n(y) \rangle) \\ &\quad < (\langle \varphi(z), \psi_1(z) \rangle, ..., \langle \varphi(z), \psi_n(z) \rangle), \end{aligned} \tag{12}$$

which is possible in exactly one of the two cases:

(a) $\varphi(y) < \varphi^*$; in this case (12) holds with any $z \in V$;

(b) $\varphi(y) = \varphi^*$, but there exists $z \in V$ such that $\psi(y) < \psi(z)$.

Obviously, (a) is equivalent to the fact that in the φ, ψ-problem (3), y has not been included in the set of solutions already at the first stage, and (b) is equivalent to the fact that y is included in the set of solutions at the first but not at the second stage of (3). So (a) and (b) together exhaust the cases when y is not a solution of (3).

The results of Subsections I and II can be summed up in the following theorem.

THEOREM 1. *Every two-stage problem of scalar–vectorial extremization (in particular, of scalar–scalar extremization) is universally reducible to a one-stage problem of vectorial (resp., scalar) extremization.*

In concluding this section we shall note that under reduction of a two-stage scalar–vectorial extremization problem to a one-stage problem, the case is possible when the resulting one-stage problem turns out to be not a vectorial but a scalar one. We shall now give the complete characterization of this case based on Lemma 1 and construction (11) of a vectorial function χ, to the extremization of which the initial φ, ψ-problem (3) may be reduced. Due to Lemma 1, a χ-problem (10) will be essentially vectorial if and only if in U there are no χ-triads, i.e., no triples $u, v, w \in U$ such that

$$\chi(u) < \chi(v), \qquad \chi(u) >\!< \chi(w), \qquad \chi(v) >\!< \chi(w). \tag{13}$$

From the definition of χ (11) it follows that the last two relations in (13) are equivalent to

$$\begin{aligned} &\varphi(u) = \varphi(v) = \varphi(w), \\ &\psi(u) >\!< \psi(w), \qquad \psi(v) >\!< \psi(w), \end{aligned} \tag{14}$$

and the first relation in (13) with first equality in (14) is equivalent to the vectorial inequality

$$\psi(u) < \psi(v). \tag{15}$$

So fulfillment of the relation system (14), (15) for some $u, v, w = U$ is necessary and sufficient for a χ-problem (10) to be essentially vectorial, i.e., irreducible to a scalar extremization problem. Hence it is also necessary and sufficient for an initial φ, ψ-problem (3) to be irreducible to a scalar problem. Speaking differently, the reducibility of a φ, ψ-problem (3) to a scalar one is equivalent to nonexistence of a situation of the form (14), (15) on the set U.

Noting that (15) together with the second line in (14) implies that the triple u, v, w is a ψ-triad, we may give the final form of the latter statement.

Addendum to Theorem 1. For a two-stage problem of scalar–vectorial extremization to be reducible to a problem of scalar extremization, it is necessary and sufficient that no subset of the set U having the same φ value on its elements (i.e., a subset of the form $U_c = \{u \in U \mid \varphi(u) = c\}$) contains a ψ-triad.

Note that due to the impossibility of ψ-triads for scalar functions ψ, this addendum automatically implies the above-mentioned universal reducibility of any scalar–scalar φ, ψ-problem to a one-stage problem.

Thus the existence of the "preliminary" scalar extremization stage before the vectorial extremization stage in any case does not make the problem "essentially more complex" than a single vectorial problem (Theorem 1). Moreover, introducing the preliminary φ-stage can even "simplify" the initial vectorial ψ-problem, transforming it into a scalar one (Addendum to Theorem 1) by "destroying" existing ψ-triads (by virtue of giving unequal φ-values to their elements).

Now let us go to clarify in what degree an extremization problem is complicated after introducing not the scalar but the vectorial preliminary extremization stage.

4. Two-Stage Extremization: General Case

In this section the main statements are given which concern the general case of vectorial–vectorial extremization (problems of type IV), and the case of vectorial–scalar extremization (type III) which turns out to be almost as complex.

Start with a problem of type III. Let $\varphi = (\varphi_1, \ldots, \varphi_n)$ be a vectorial and ψ a scalar function on U. One might attempt, by analogy with a problem of type II, to design a "lexicographic" vectorial function χ with components

$\chi_i = \langle \varphi_i, \psi \rangle$, $i = 1,..., n$. But it is easy to verify that the initial two-stage problem (3) definitely is not reducible to the extremization of the function χ in the nontrivial case when for at least one pair of elements, $p, q \in U$,

$$\varphi(p) >\!\!< \varphi(q) \qquad \text{but} \qquad \psi(p) \neq \psi(q). \tag{16}$$

Really, in this case for the two-stage φ, ψ-problem of extremization on $X = \{p, q\}$ the set Y evidently contains just one of the elements p or q. But for the respective one-stage χ-problem we have $\chi(p) >\!\!< \chi(q)$ and hence $Y = \{p, q\}$ on the same $X = \{p, q\}$.

Attempts at constructing any other function χ, scalar or vectorial, to reduce a two-stage φ, ψ-problem to a one-stage χ-problem are doomed to failure. Consider the triple $u, v, w \in U$ such that

$$\begin{gathered} \varphi(u) < \varphi(v), \qquad \varphi(u) >\!\!< \varphi(w), \qquad \varphi(v) >\!\!< \varphi(w), \\ \psi(u) > \psi(w) \geqslant \psi(v) \end{gathered} \tag{17}$$

(here the first line in (17) means that u, v, w is a φ-triad). Then for the two-stage φ, ψ-problem on $X = \{u, v, w\}$ the solution set Y surely includes element w. But on $X = \{u, w\}$ the respective solution set Y' contains the single element u (i.e., $Y' = \{u\}$) and does not include w. It demonstrates the violation of the property of "rational choice," the *heredity property*[2] which may be formulated in the following way: if $w \in X' \subseteq X$ and $w \in Y$ where Y is the choice from X, then $w \in Y'$ holds where Y' is the choice from X'. It is easy to see that the heredity property must be kept on solution sets of any one-stage extremization problem (considered as a mass problem of choice, following the Introduction). Therefore the above-stated φ, ψ-problem on the set U including the triple u, v, w from (17) definitely cannot be reduced to a one-stage problem of extremization, neither scalar nor vectorial.

Thus all problems of vectorial–scalar extremization and, even more so, all problems of vectorial–vectorial extremization of the general type must be broken up into definitely nonempty classes: problems which do not admit universal reduction to the one-stage problem of scalar or vectorial extremization, and problems which do admit such a reduction. We shall give in this section the formulations of conditions which discriminate these classes of problems. The proofs will be given in the next section.

We introduce the necessary definitions.

DEFINITION 3. Let φ and ψ be two vectorial functions on U and let a triple $u, v, w \in U$ be a φ-triad (8). We shall call it a *φ, ψ-inconsistent triad of*

[2] See, e.g., [1]. Such a property also used to be called the Chernoff condition or α-condition (after Sen [4]).

the 1st, 2nd or 3rd kind (in short, a φ, ψ-1-, φ, ψ-2-, or φ, ψ-3-*triad*) if, respectively,

$$\begin{aligned} &(1) \quad \psi(u) > \psi(w), \qquad \psi(v) \not> \psi(w); \\ &(2) \quad \psi(u) >\!< \psi(w), \qquad \psi(v) < \psi(w); \\ &(3) \quad \psi(u) >\!< \psi(w), \qquad \psi(v) >\!< \psi(w). \end{aligned} \tag{18}$$

Remark. In Definition 2 as usual it is not forbidden to take as vectorial functions φ, ψ their particular case, scalar functions. But we note that if the first function, φ, is really scalar, then φ-triads and, even more so, φ, ψ-1-, φ, ψ-2-, and φ, ψ-3-triads surely do not exist.

We now formulate the main theorem of this work.

THEOREM 2. *For a vectorial–vectorial extremization φ, ψ-problem to be universally reducible to a vectorial (or, moreover, to a scalar) one-stage extremization problem, it is necessary and sufficient that φ, ψ-1- and φ, ψ-2-triads be absent in U (respectively, any φ, ψ-1-, φ, ψ-2-, φ, ψ-3-, and also ψ, φ-3-triads were absent in U).*

Remark. Theorem 2 covers the general vectorial–vectorial case of the two-stage extremization problem (type IV). Surely it implies the statements concerning reducibility of more particular problems, of types I and II, which have been considered above, and also of type III. Being applied to problems of type III, Theorem 2 can be specified in the following way. In the formulation of Theorem 2, φ, ψ-inconsistent triads of the 1st, 2nd, and 3rd kind are characterized as φ-triads satisfying the conditions, respectively,

$$\begin{aligned} &(1) \quad \psi(u) > \psi(w) \geqslant \psi(v); \\ &(2) \quad \psi(u) = \psi(w) > \psi(v); \\ &(3) \quad \psi(u) = \psi(w) = \psi(v). \end{aligned} \tag{19}$$

The additional requirement of the absence of ψ, φ-3-triads (in the list of conditions for reducibility to a scalar problem) here may be omitted, because due to scalarity of ψ it is fulfilled automatically. Note that just the φ-triad with additional condition (19.1) has been considered in this section as a counterexample for the reducibility of a φ, ψ-problem of type III to the one-stage problem. Furthermore, for problems of type I and II, Theorem 2 immediately gives Theorem 1 and the Addendum to it.

Indeed, under scalarity of φ no φ, ψ-inconsistent triads exist, hence reducibility of a problem of type II (in particular, of type I) to a vectorial problem is always guaranteed. Moreover, under scalarity of ψ no ψ, φ-3-triads exist, hence a problem of type I is always reducible to a scalar problem. It results just in Theorem 1. Finally, reducibility of an arbitrary problem of type II to a scalar problem is equivalent to the absence of ψ, φ-3-triads (because any φ, ψ-triads are definitely absent). But relations (14), (15), whose fulfillment is excluded by the Addendum to Theorem 1, just compose the definition of a ψ, φ-3-triad. So the results from Section 3 are essentially included in the generalizing Theorem 2 as particular cases.

5. Proof of the General Theorem on the Vectorial–Vectorial Problem

The proof given below is founded upon a number of statements concerning binary relations and kindred notions in the theory of choice and decision making.

A. *Representation if Binary Relations of Ordering by Numerical Functions*

Let on a set U a binary relation $\succ$ be given which is asymmetrical, i.e., $u \succ v \Leftrightarrow (\text{not } u \prec v)$; we shall further call $\succ$ a *superiority relation*. For the given relation $\succ$, we construct the associated *indifference relation* denoted by the symbol $\sim$ and defined by

$$u \sim v \Leftrightarrow (\text{not } u \succ v \text{ and not } u \prec v).$$

A relation $\succ$ on U is called a *partial order* if it is transitive, i.e., $(u \succ v$ and $v \succ w) \Rightarrow u \succ w$, and a *weak order* if, moreover, the respective relation $\sim$ is also transitive (here orders are considered in the "strict" version according to the presumed asymmetry of $\succ$).

Definition 4. Let a vectorial (possibly scalar) function f and a relation $\succ$ be given. We say that f is *consistent with* $\succ$ if for every $u, v \in U$:

$$u \succ v \Rightarrow f(u) > f(v), \tag{20}$$

and that f *represents* $\succ$ if, moreover,

$$u \succ v \Leftrightarrow f(u) > f(v), \tag{21}$$

where $>$ is the symbol of vectorial inequality in the Paretian sense (7).

This Definition directly implies

Lemma 2. *Every scalar (or vectorial) function f on U represents some relation of weak (resp., partial) order $\succ$ on U; moreover, the relation represented thereby is unique.*

The next two lemmata are as a matter of fact the versions of the well-known Szpilrajn theorem and the Dushnik and Miller theorem [3, 5] (concerning imbedding of partial orders into linear ones) which are expressed in terms of representing functions.

LEMMA 3. *For any partial (also including weak) order $\succ$ on U there exists a scalar function f on U consistent with $\succ$.*

LEMMA 4. *For any weak (or partial) order $\succ$ on U there exists a scalar (resp., vectorial) function f on U which represents $\succ$.*

We shall give here independent "constructive" proofs of Lemmata 3 and 4 using a simple iterative procedure in the spirit of R. Bellman's dynamic programming, for building up functions f to be found. Let a superiority relation $\succ$ and the associated indifference relation $\sim$ on U be given. Consider the iterative procedure

$$f^{t+1}(x) = \max \{f^t(x); \max_{s \prec x} (f^t(s) + 1)\} \tag{22}$$

(we mean that here $x \in U$, $s \in U$, $t = 0, 1,\ldots$, and that in the case of the absence of $s \in U$ such that $s \prec x$, only the first term remains in the outer braces in (22)).

Take initial values of f:

$$f^0(u) = 0, \qquad \text{for all } u \in U. \tag{23}$$

It follows from acyclicity of $\succ$ that in U there exists at least one minimal element, i.e., element u such that for it there are no elements v such that $u \succ v$. For every minimal element u in U obviously $f^t(u) \equiv 0$, for all $t = 0, 1,\ldots$.

Furthermore, the definition of the procedure (22) in virtue of acyclicity and transitivity of $\succ$ implies the next properties:

(a) Values $f^t(x)$ for every fixed x do not decrease with t increasing.

(b) Positiveness of the value $f^t(x) = k$ for some x and t is equivalent to existence of a chain $x \succ p \succ q \succ \cdots \succ u$ in U, where u is a minimal element of U and the length of the chain (the number of elements $x, p, q,\ldots, u$) is equal to k.

(c) It follows from Property (b) that

$$f^t(x) < N \qquad \text{for all } t = 0, 1,\ldots, \qquad \text{and} \qquad x \in U,$$

where $N = |U|$.

(d) It follows from Properties (a) and (b) that after a finite number of steps, T, the procedure (22) stabilizes on some values $f^*(x)$, i.e.,

$$f^{t+1}(x) = f^t(x) = f^*(x), \qquad \text{for all } t \geqslant T, \quad x \in U. \tag{24}$$

(e) It follows from Properties (b) and (c) that for (24) the following estimates are true: $T < N$, and for all $x \in U$

$$f^*(x) < N. \tag{25}$$

(f) It follows from Property (d) that the function f^* on U is consistent with the relation $\succ$ (it is sufficient to consider (22) with (24)).

Property (f) yields the statement of Lemma 3. For Lemma 4 to be proved for the case when $\succ$ is a weak order, it is suffcient now to verify that in this case the function f^* so constructed is not only consistent with $\succ$ but, moreover, does represent $\succ$. To this end we need to show that if $u \sim v$ then $f^*(u) = f^*(v)$. Let $u \sim v$. Properties of the weak order imply

$$s \prec u \Leftrightarrow s \prec v, \qquad \forall s \in U.$$

Taking into account the form of procedure (22) and initial data (23) the above equivalence implies $f^t(u) \equiv f^t(v)$, for $u \sim v$ with all $t = 0, 1, \ldots$, hence $f^*(u) = f^*(v)$.

Finally we shall prove the Lemma for the case when $\succ$ is a partial order. With that end in view we consider the procedure (22) under special initial conditions different from (23). Let elements of U be numbered: $U = \{u_1, u_2, \ldots, u_n\}$. Consider N different sets of initial data specifying in the ith set ($i = 1, \ldots, N$)

$$f_i^0(x) = \begin{cases} N, & \text{for } x = u_i, \\ 0, & \text{for } x \neq u_i. \end{cases} \tag{26}$$

Applying now the procedure (22) with each of these N sets of initial data, we shall get N function sequences $\{f_i^t\}_{t=0,1,\ldots}$ ($i = 1, \ldots, N$) built by the procedure. Each sequence will possess properties similar to (and partially coincident with) Properties (a)–(f) of the sequence $\{f_i^t\}$ generated by the procedure (22) under zero initial data (23). For convenience of describing properties of the new sequences $\{f_i^t\}$ determined from the ith set of initial data (26), we partition the set U into two subsets

$$U_i^+ = \{u \in U \mid u \succ u_i \text{ or } u = u_i\} \qquad \text{and} \qquad U_i^- = U \setminus U_i^+.$$

Consider the behavior of value sequences $\{f_i^t(u)\}_{t=0,1,\ldots}$ for $u \in U_i^-$ and for $u \in U_i^+$ separately. It is not difficult to see that two respective subsets $\{f_i^t(v)\}$, $v \in U_i^-$, and $\{f_i^t(w)\}$, $w \in U_i^+$, can be constructed by independent application of procedure (22) to subsets U_i^- and U_i^+ separately taken in the role of U in (22) under initial data (23) or (26), respectively. Really, let us consider the cases of U_i^- and U_i^+ taken separately.

(1) *Case of* U_i^-. To begin with we note that by the construction of sets U_i^+ and U_i^-, due to the transitivity of $\succ$, the following holds:

$$\text{If } v \in U_i^- \text{ and } w \in U_i^+ \text{, then it is impossible that } v \succ w. \tag{27}$$

Hence the procedure (22) being applied to the total set U under initial data (26) works on the subset U_i^- independently of U_i^+, i.e., generates the same sequences $\{f_i^t(v)\}_{t=0,1,\ldots}$, $v \in U_i^-$, as when applied only to U_i^- under zero initial data $f_i^0(v)$, $v \in U_i^-$. Therefore the subset $\{f_i^t(v)\}_{t=0,1,\ldots}$, $v \in U_i^-$, possesses all of the above-stated properties (a)–(f), including finite convergence to a subset of values $f_i^*(v)$, $v \in U_i^-$, which is consistent with $\succ$ on U_i^- and is such that

$$f_i^*(v) < N, \qquad v \in U_i^-. \tag{28}$$

(Here the estimate N can actually be made more precise, up to $N_i^- = |U_i^-| < N = |U|$.)

(2) *Case of* U_i^+. Note that the procedure (22) under initial data (26) for $t = 1$ yields

$$f_i^1(u_i) = N; \qquad f_i^1(u) = N + 1, \qquad \text{for} \quad u \in U_i^+, \quad u \neq u_i. \tag{29}$$

Hence due to monotonic nondecreasing of $f_i^t(u)$ in t, we receive for all $t \geqslant 1$:

$$f_i^t(u_i) \equiv N; \qquad f_i^t(u) \geqslant N + 1, \qquad \text{for} \quad u \in U_i^+, \quad u \neq u_i. \tag{30}$$

Comparing (30) with (28), we conclude that the procedure (22) works over subset U_i^+ independently on U_i^-, i.e., generates the same sequences $\{f_i^t(w)\}_{t=0,1,\ldots}$, $w \in U_i^+$, as when applied only to U_i^+ under a subset of initial data (26) for $x \in U_i^+$. On the other hand, let us apply the procedure (22) to U_i^+ under zero initial data: $f_i^0(w) = 0$, $w \in U_i^+$. Then because u_i obviously is the unique minimal element in U_i^+, we shall get for $t = 1$:

$$f_i^1(u_i) = 0; \qquad f_i^1(u) = 1, \qquad \text{for} \quad u \in U_i^+, \quad u \neq u_i. \tag{31}$$

Comparing (31) with (29), we conclude that the procedure (22) on U_i^+ under initial data set (26) generates values $f_i^t(w)$, $t = 1, 2, \ldots$, which exceed exactly by N the corresponding values given under zero initial data. Latter values as it has been pointed out earlier must possess all Properties (a)–(f). Hence the procedure (22) ensures finite convergence of the sequence subset

$\{f_i^t(w)\}_{t=0,1,\ldots}$, $x \in U_i^+$, to the value subset $f_i^*(w)$, $w \in U_i^+$, consistent with $\succ$ on U_i^+ and such that

$$N \leqslant f_i^*(w) < 2N, \qquad \text{for all } w \in U_i^+. \tag{32}$$

Finally, it is easy to see that united set of values $f_i^*(u)$, $u \in U = U_i^- \cup U_i^+$, yields the function f_i^* on U consistent with $\succ$ on the overall set U (due to (27), (28), and (32)).

Let us now construct a vectorial function f^* not only consistent with but representing the relation $\succ$ on U. Note that the above-constructed scalar function f_i^* possesses the following property:

$$\text{If} \quad u \neq u_i \quad \text{and} \quad u \not\succ u_i, \quad \text{then} \quad f_i^*(u) < f_i^*(u_i) \tag{33}$$

(it follows from estimates (28) and (32)). Letting $f^* = (f_1^*, \ldots, f_N^*)$, we get f^* consistent with $\succ$ on U such that (due to (33)), for every $u, v \in U$, we have the following additional property:

$$\text{If} \quad u \not\succ v, \qquad \text{then} \qquad f^*(u) \not> f^*(v). \tag{34}$$

Condition (34) together with consistency of f^* with $\succ$ implies that f^* does represent $\succ$ on U. This ends the proof of Lemma 4 for the general vectorial case.

Remark. This constructive proof of the existence of a vectorial function f^* representing $\succ$ on U simultaneously gives an upper bound for its dimensionality: it is sufficient to take f^* with no more than $N = |U|$ scalar components.

B. *Problems of Extremization under Binary Relations*

Let $\succ$ be a superiority relation on U (i.e., an asymmetrical binary relation). We call a *problem of extremization under relation* $\succ$ (in short, a $\succ$*-problem*) the following problem of selection of the subset Y from a set X defined as

$$Y = \{y \in X \mid \text{there is no } x \in X \text{ such that } x \succ y\}. \tag{35}$$

Every $\succ$-problem being considered as a mass problem (over all $X \in \mathscr{X}$) represents a special type of choice problem $X \to Y$, $X \in \mathscr{X}$. The main question interesting us herein concerns the reducibility of a choice problem to the problem of extremization of a function (f-problem). It is convenient to reduce the solution of this question to the two steps: (1) applying a criterion of reducibility of a choice problem to a $\succ$-problem, and (2) applying a criterion of reducibility of a $\succ$-problem to an f-problem. Let us begin with the 1st step.

DEFINITION 5. For an arbitrary choice problem $X \to Y$ on $\mathscr{X} = 2^U \setminus \{\varnothing\}$, we shall call the *relation of pairwise-revealed superiority* the binary relation $\check{P}$ on U defined as

$$u\check{P}v \Leftrightarrow [u \text{ but not } v \text{ is chosen from the pair } \{u, v\}] \tag{36}$$

for every $u, v \in U$.

Remark. Relation $\check{P}$ in virtue of Definition 5 surely is asymmetrical, i.e., it is really a superiority relation.

LEMMA 5. *If a choice problem $X \to Y$ on $\mathscr{X} = 2^U \setminus \{\varnothing\}$ is reducible to a problem of extremization under some superiority relation $\succ$, then $\succ$ coincides with the pairwise-revealed superiority relation $\check{P}$ for this problem.*

Proof of the lemma is reduced to comparing the above definition (36) with the definition of choice (35) for the case $X = \{u, v\}$.

Lemma 5 implies directly the following criterion of reducibility of a choice problem to a $\succ$-problem.

LEMMA 6. *For a choice problem $X \to Y$ on $\mathscr{X} = 2^U \setminus \{\varnothing\}$ to be reducible to a $\succ$-problem, it is necessary and sufficient that for every $X \in \mathscr{X}$ the following is true*:

$$Y = \{y \in X \mid \text{there is no } x \in X \text{ such that } x\check{P}y\}. \tag{37}$$

Equation (37) will be called the *Condorcet Principle.* The essence of the Condorcet Principle is the requirement that a given choice problem $X \to Y$ be reducible to the problem of extremization under its relation of pairwise-revealed superiority, i.e., to the $\succ$-problem with $\succ = \check{P}$. Lemma 6 says that this requirement is not only sufficient (which is trivial) but also necessary (which is declared by Lemma 6) for reducibility to a $\succ$-problem.

Let us go to the 2nd step, to the statements concerning mutual reducibility of $\succ$-problems to f-problems.

LEMMA 7. *If a superiority relation $\succ$ on U is represented by a function f, then the problem of extremization under the relation $\succ$ and the problem of extremization of the function f on $\mathscr{X} = 2^U \setminus \{\varnothing\}$ are mutually reducible, one to another.*

Proof of the Lemma is given by direct replacement of the relation $x \succ y$ by $f(x) > f(y)$ in the formulation of the $\succ$-problem.

Remark. Lemma 7 admits a converse: if an f-problem and a $\succ$-problem on $\mathscr{X} = 2^U \setminus \{\varnothing\}$ are mutually reducible, then f represents $\succ$. It follows directly from Lemma 5 applied to the f-problem because for this problem

$$x\check{P}y \Leftrightarrow [f(x) > f(y)].$$

Lemma 7 opens the path of construction of direct counterparts of Lemmata 2 and 4 in terms of extremization problems:

LEMMA 8. *Each problem of extremization of a scalar (or vectorial) function f on U is reducible to the problem of extremization under a superiority relation $\succ$ which is a weak (resp., partial) order on U.*

LEMMA 9. *Each problem of extremization under a superiority relation $\succ$ which is a weak (or partial) order on U is reducible to a problem of extremization of a scalar (resp., vectorial) function f on U.*

Now we can synthesize the two steps of the reasoning and formulate the next lemma.

LEMMA 10. *For a choice problem $X \to Y$ on $\mathscr{X} = 2^U \setminus \{\varnothing\}$ to be reducible to the problem of extremization of a scalar (or vectorial) function f, it is necessary and sufficient that the following two conditions be fulfilled:*

(1) *the given problem satisfies the Condorcet Principle;*

(2) *the pairwise-revealed superiority relation $\check{P}$ is a weak (resp., partial) order.*

Proof. (a) Necessity: Let the given choice problem be reducible to the f-problem with a scalar (or vectorial) function f. Then in virtue of Lemma 8, it is reducible also to the $\succ$-problem with a relation $\succ$ being a weak (resp., partial) order. The latter in virtue of Lemma 6 implies satisfying the Condorcet Principle for the initial problem (condition(1)). Moreover, in virtue of Lemma 5, its relation $\check{P}$ must coincide with the relation $\succ$, and hence $\check{P}$ must be a weak (resp., partial) order (condition (2)).

(b) Sufficiency: Let the given choice problem satisfy conditions (1) and (2) of the lemma. Then due to condition (1), in virtue of Lemma 6 the given problem is reducible to the problem of extremization under the relation $\succ = \check{P}$, which is by condition(2), a weak (or partial) order. But then in virtue of Lemma 9 the initial problem is reducible also to the problem of extremization of a scalar (resp., vectorial) function f. Q.E.D.

Remark. Now a proof of Lemma 1 can be obtained as a simple corollary from Lemma 10. Really, in virtue of Lemma 10 for reducibility of a vectorial φ-problem to a scalar problem, it is necessary and sufficient that the superiority relation $\check{P}$ having form $x\check{P}y \Leftrightarrow \varphi(x) > \varphi(x) > \varphi(y)$ is a weak order. It is equivalent to transitivity of the respective indifference relation $\check{I}$ of the form $x\check{I}y \Leftrightarrow \varphi(x) >< \varphi(y)$ which is just equivalent to the absence of φ-triads in U.

C. *Proof of Theorem* 2

Apply the routine of two-step analysis given in the form of the two conditions in Lemma 10 to the analysis of reducibility of a vector–vectorial φ, ψ-problem to some one-stage vectorial or scalar problem.

(1) *Conditions of satisfying the Condorcet Principle for the* φ, ψ-*problem.* To begin with let us write the pairwise-revealed superiority relation $\check{P}$ for the φ, ψ-problem (3):

$$u\check{P}v \Leftrightarrow [\varphi(u) > \varphi(v) \text{ or } (\varphi(u) >< \varphi(v) \text{ and } \psi(u) > \psi(v))]. \tag{38}$$

The testing of the Condorcet Principle (37) is reduced to the testing of the two statements which together form the Principle. These two statements (for any $X \in \mathscr{X}$ and for the choice Y from X), are, respectively:

$$\text{If } y \in X \text{ and if there is no } x \in X \text{ such that } x\check{P}y, \text{ then } y \in Y \tag{39}$$

(*Direct Condorcet Condition*), and:

$$\text{If } y \in Y, \text{ then there is no } x \in X \text{ such that } x\check{P}y \tag{40}$$

(*Converse Condorcet Condition*).

Let us first examine the Direct Condorcet Condition (39). Assume that in a φ, ψ-problem $y \notin Y$ holds for some $X \in \mathscr{X}$ and for some $y \in X$. The form of two-stage problem (3)–(4) shows that it is possible in one of two cases: (a) there exists $z \in X$ such that $\varphi(z) > \varphi(y)$, and then $y \notin V$; or (b) $y \in V$, but there exists $z' \in X$ such that $z' \in V$ and $\psi(z') > \psi(y)$. In the case (b) there necessarily holds $\varphi(z') >< \varphi(y)$. It is easy to see that in the case (a), we have $z\check{P}y$ and in the case (b), we have $z'\check{P}y$. Therefore, if for y and X with $y \in X$, $x\check{P}y$ holds for no $x \in X$, then necessarily $y \in Y$. Hence any φ, ψ-problem does satisfy the Direct Condorcet Condition (39).

Consider now the Converse Condorcet Condition (40). Assume that the given φ, ψ-problem violates this condition, viz, there exist $X \in \mathscr{X}$ and $x, y \in X$ such that $y \in Y$ but $x\check{P}y$. From $y \in Y$ and $x \in X$ follows

$$\varphi(x) \not> \varphi(y). \tag{41}$$

In virtue of $x\check{P}y$ then by (41)

$$\varphi(x) >< \varphi(y) \qquad \text{and} \qquad \psi(x) > \psi(y). \tag{42}$$

Assumption $y \in Y$ implies $x \notin V$ by (42). Therefore there exists a third element $z \in X$ such that

$$\varphi(z) > \varphi(x). \tag{43}$$

On the other hand, in virtue of $y \in Y$, $z \in X$, we have

$$\varphi(z) \not> \varphi(y). \tag{44}$$

Furthermore, the relation $\varphi(y) > \varphi(z)$ is also impossible because in this case in virtue of (25) and transitivity of $>$, we would obtain $\varphi(y) > \varphi(x)$, contrary to (42). Hence $\varphi(y) \not> \varphi(z)$, which being combined with (44) yields

$$\varphi(y) >< \varphi(z). \tag{45}$$

Now if for the given z there exists one more element $z' \in X$ such that $\varphi(z') > \varphi(z)$, then it together with (43) yields $\varphi(z') > \varphi(x)$. If, furthermore, there exists $z'' \in X$ such that $\varphi(z'') > \varphi(z')$, then in a similar way $\varphi(z'') > \varphi(x)$, etc. Due to the finiteness of X and the acyclicity of $>$, there always exists an element in X, we preserve the notation z for it, such that (43) holds and there is no $s \in X$ such that $\varphi(s) > \varphi(z)$, and hence $z \in V$. But then for $y \in Y \subseteq V$, we obtain

$$\psi(z) \not> \psi(y). \tag{46}$$

Relations (42), (43), (44), and (46) together imply that the triple x, y, z is a φ, ψ-1-triad (letting $u = x$, $v = z$, $w = y$ in the definition (8), (18.1)). Conversely, if there exists a φ, ψ-1-triad u, v, w in U, then letting $X = \{u, v, w\}$, we obtain $w \in Y$, $v \in X$, but $v\check{P}w$, contrary to the Converse Condorcet Condition. Therefore the Converse Condorcet Condition is not fulfilled if and only if there exists a φ, ψ-1-triad in U.

So we have proved the following Lemma.

LEMMA 11. *For a vectorial–vectorial φ, ψ-problem to satisfy the Condorcet Principle, it is necessary and sufficient that φ, ψ-1-triads were absent in U.*

(2) *Conditions of order for relation $\check{P}$ in the φ, ψ-problem.* Now we shall find conditions for transitivity of a superiority relation $\succ = \check{P}$ and also of a corresponding indifference relation $\sim$ which will be denoted here by $\check{I}$.

LEMMA 12. *Let a vectorial–vectorial φ, ψ-problem satisfy the Condorcet Principle, and let $\check{P}$ be the pairwise-revealed superiority relation for this problem and $\check{I}$ the corresponding indifference relation. Then*:

(a) *for transitivity of $\check{P}$, it is necessary and sufficient that φ, ψ-2-triads were absent in U*;

(b) *for transitivity of $\check{I}$, it is necessary and sufficient that φ, ψ-3- and ψ, φ-3-triads were absent in U.*

Proof of lemma. (a) Nontransitivity of $\check{P}$ would imply that there is a triple x, y, z of elements in U, such that

$$x\check{P}y, \qquad y\check{P}z, \qquad \text{but not } x\check{P}z. \tag{47}$$

There are three possible systems of relations between values φ and ψ on the triple x, y, z which according to (38) give (47):

$$\begin{cases} \varphi(x) > \varphi(y), & \varphi(y) >\!< \varphi(z), \quad \varphi(x) >\!< \varphi(z), \\ & \psi(y) > \psi(z), \quad \psi(x)) \not> \psi(z), \end{cases} \tag{48}$$

and

$$\begin{cases} \varphi(x) >\!< \varphi(y), & \varphi(y) > \varphi(z), \quad \varphi(x) >\!< \varphi(z), \\ \psi(x) > \psi(y), & \qquad\qquad\qquad\; \psi(x) \not> \psi(z), \end{cases} \tag{49}$$

and also

$$\begin{cases} \varphi(x) >\!< \varphi(y), & \varphi(y) >\!< \varphi(z), \quad \varphi(x) < \varphi(z), \\ \psi(x) > \psi(y), & \psi(y) > \psi(z). \end{cases} \tag{50}$$

(No other admissible system of relations, with transitivity of $>$ in mind, can give (47).) In the case (48) the triple x, y, z forms a φ, ψ-1-triad (with $u = y$, $v = x$, $w = z$). In the case (49), it forms a φ, ψ-1-triad or a φ, ψ-2-triad (with $u = z$, $v = y$, $w = x$) depending on whether $\psi(x) < \psi(z)$ or $\psi(x) >\!< \psi(z)$. Finally, in the case (50) it forms again a φ-1-triad (with $u = x$, $v = z$, $w = y$). The considered φ, ψ-problem was assumed to satisfy the Condorcet Principle, hence in virtue of Lemma 11 any φ, ψ-1-triads are absent in U. Therefore, nontransitivity of $\check{P}$ is possible only under existence of some φ, ψ-2-triad in U.

Conversely, every φ, ψ-2-triad u, v, w (8), (18.3) yields

$$v\check{P}u, \qquad u\check{P}w, \qquad \text{but not } v\check{P}w,$$

i.e., a violation of the transitivity of $\check{P}$. Part (a) of Lemma 12 is proved.

(b) Nontransitivity of $\check{I}$ implies that in U there is a situation of the form

$$x\check{I}y, \qquad y\check{I}z, \qquad \text{but } x\check{P}z. \tag{51}$$

Two systems of relations implementing the situation (15) are possible:

$$\begin{cases} \varphi(x) \gtrless \varphi(y), & \varphi(y) \gtrless \varphi(z), \quad \varphi(x) > \varphi(z), \\ \psi(x) \gtrless \psi(y), & \psi(y) \gtrless \psi(z), \end{cases} \tag{52}$$

and

$$\begin{cases} \varphi(x) \gtrless \varphi(y), & \varphi(y) \gtrless \varphi(z), \quad \varphi(x) \gtrless \varphi(z), \\ \psi(x) \gtrless \psi(y), & \psi(y) \gtrless \psi(z), \quad \psi(x) > \psi(z). \end{cases} \tag{53}$$

In the case (52) the triple x, y, z forms a φ, ψ-3-triad (with $u = z$, $v = x$, $w = y$), and in the case (53) it forms a ψ, φ-3-triad (with the same u, v, w).

Conversely, every φ, ψ-3-triad u, v, w (8), (18.3) yields $v\check{I}w$, $w\check{I}u$, but $v\check{P}u$, which is a violation of the transitivity of $\check{I}$. Furthermore, every ψ, φ-3-triad, i.e., a triple $u, v, w \in U$ such that

$$\begin{cases} \psi(u) < \psi(v), & \psi(u) \gtrless \psi(w), \quad \psi(v) \gtrless \psi(w), \\ & \varphi(u) \gtrless \varphi(w), \quad \varphi(v) \gtrless \varphi(w), \end{cases}$$

yields $u\check{I}w$, $v\check{I}w$, and either $u\check{P}v$ (if $\varphi(u) > \varphi(v)$) or $v\check{P}u$ (if $\varphi(u) \not> \varphi(v)$), which is a violation of the transitivity of $\check{I}$ again. This completes the proof of Lemma 12.

Combining Lemmata 11, 12 with 10, we obtain the proof of Theorem 2.

6. Conditions for the Elimination of One Stage in the Two-Stage Problem

Pose the question of the reducibility of a two-stage, generally vector-vectorial, φ, ψ-problem not only to some one-stage problem of extremization of some function f, but more definitely, to the problem of extremization of just the same function $f = \varphi$ or $f = \psi$ which is present in one of two stages of the initial problem. Speaking in a different way, the question is: under what conditions is it possible to discard, i.e., simply to "throw out," one of the stages in the two-stage problem without changing its solution set?

It is clear that these conditions must be surely more strict than the general condition of the reducibility of a two-stage φ, ψ-problem to a one-stage problem of extremization of some arbitrary function f. This latter condition has been obtained in Theorem 2 for the general case of the vectorial–vectorial φ, ψ-problem in the form of prohibition of special three-element structural formations in U, viz, φ, ψ-1- and φ, ψ-2-inconsistent triads. It turns out that the sought-after more strict conditions of reducibility can be formulated in the form of stronger prohibitions. Namely, we must prohibit certain two-element structural formations, "dyads," implicitly contained in the above-mentioned triads.

DEFINITION 6. A pair of elements $x, y \in U$ will be called a *φ, ψ-inconsistent dyad* if

$$\varphi(x) < \varphi(y), \qquad \psi(x) \nless \psi(y), \tag{54}$$

and a *φ, ψ-noncoherent* dyad if

$$\varphi(x) < \varphi(y) \qquad \text{but} \qquad \psi(x) >\!< \psi(y). \tag{55}$$

Note that any φ, ψ-noncoherent dyad is a particular case of a φ, ψ-inconsistent dyad.

Definitions 6 and 3 immediately imply:

LEMMA 13. *Every φ, ψ-1- and every φ, ψ-2-inconsistent triad contains a φ, ψ-inconsistent dyad and a ψ, φ-noncoherent dyad.*

Observe that in this Lemma and in the sequel we consider φ, ψ-inconsistent but ψ, φ-noncoherent (with reverse order of φ and ψ!) dyads.

THEOREM 3. *For a two-stage vectorial–vectorial φ, ψ-problem to be reducible to its first (or second) stage, i.e., to the problem of extremization of the vectorial function φ (respectively, ψ), it is necessary and sufficient that in U any ψ, φ-noncoherent dyads (respectively, φ, ψ-inconsistent dyads) be absent.*

Before proving the Theorem, we note that the condition of absence of φ, ψ-inconsistent dyads in U can be obviously presented in the following equivalent form:

$$\varphi(x) < \varphi(y) \Rightarrow \psi(x) < \psi(y), \qquad \text{for all } x, y \in U \tag{56}$$

(a condition of *φ, ψ-consistency*). Similarly, let us consider the condition of absence of ψ, φ-noncoherent dyads, i.e., of pairs $u, v \in U$ such that $\psi(u) < \psi(v)$ but $\varphi(u) >\!< \varphi(v)$. It is easy to see that this condition can be presented in the following equivalent form: $\varphi(u) >\!< \varphi(v) \Rightarrow \psi(u) \not< \psi(v)$, for all $u, v \in U$, which is also equivalent to

$$\varphi(x) >\!< \varphi(y) \Rightarrow \psi(x) >\!< \psi(y), \qquad \text{for all } x, y \in U \tag{57}$$

(condition of ψ, *φ-coherence*). Hence for proving Theorem 3, it is sufficient to prove the following equivalent statement:

For the reducibility of a given φ, ψ-problem to the φ-problem or to the ψ-problem, it is necessary and sufficient to meet the ψ, φ-coherence condition (57) or, respectively, the φ, ψ-consistency condition (56).

Remark. Note that the φ, ψ-consistency condition (56) coincides with the requirement of consistency (in the sense of Def. 4) of the function ψ with the relation $\succ$ represented by the function φ.

Now consider two versions of superiority relations $\succ$ represented by functions φ and ψ:

$$u \succ v \Leftrightarrow \varphi(u) > \varphi(v) \tag{58}$$

and

$$u \succ v \Leftrightarrow \psi(u) > \psi(v), \tag{59}$$

respectively. The consideration of the expressions (58) and (59) together with the expression (38) defining the pairwise-revealed superiority relation $\check{P}$ in the φ, ψ-problem yields directly the following lemma.

LEMMA 14. *For the relation $\check{P}$ from* (38) *to coincide on U with the relation $\succ$ from* (58) *or from* (59), *it is necessary and sufficient to meet the ψ, φ-coherence condition* (57) *or, respectively, the φ, ψ-consistency condition* (56).

Proof of Theorem 3. Assume that a given φ, ψ-problem is reduced to the φ-problem (or to the ψ-problem) with the function φ (or ψ) just the same as is present in the initial φ, ψ-problem. But each φ-problem (resp., ψ-problem) in its own turn is reducible (due to Lemma 7) to the problem of extremization under the relation $\succ$, represented by the function φ (resp., ψ), i.e., defined from (58) (resp., (59)). Therefore, the initial φ, ψ-problem must be reducible to the problem of extremization under the relation $\succ$ from (58) (resp., (59)). But then due to Lemma 5 the relation $\succ$ must coincide with the relation $\check{P}$ for the initial φ, ψ-problem. Hence due to Lemma 14 the condition (57) (resp., (56)) necessarily is satisfied.

Conversely, let the condition (57) or (56) be satisfied, i.e., equivalently, the condition of absence of ψ, φ-noncoherent or φ, ψ-inconsistent dyads in U be fulfilled. Then due to Lemma 13 in every case φ, ψ-1- and φ, ψ-2-

inconsistent triads must be absent in U. Hence by Theorem 2 the initial φ, ψ-problem must be reducible to a problem of extremization of some (generally vectorial) function f. Therefore, according to Lemma 10, the φ, ψ-problem must satisfy the Condorcet Principle, i.e., it must be reducible to the problem of extremization under the relation $\check{P}$ of the form (38). But in virtue of condition (57) (or, respectively, (56)) by Lemma 14 the relation $\check{P}$ coincides with the relation $\succ$ from (58) (resp., from (59)), i.e., with the relation represented by the function φ (resp., by ψ). Hence in virtue of Lemma 7 the given φ, ψ-problem will be reducible to the φ-problem or to the ψ-problem, respectively.

The Theorem is proved.

7. An Illustrative Example

Consider a simple mechanism for generating a two-stage vectorial–scalar extremization. Let an n-dimensional criterial function $\varphi(x) = (\varphi_1(x),..., \varphi_n(x))$ on a set U be given, and let a fixed "ideal point" ω in the n-dimensional space of vectors of criterial values φ also be given. Suppose that a choice of elements from an admissible set $X \in \mathscr{X}$ is implemented in two stages. At the first stage the set V of Pareto-optimal elements of X by φ is selected. The set V corresponds to the Pareto subset $\varphi(V)$ of the admissible set $\varphi(X)$ in φ-space. At the second stage those elements y are selected from V which are represented in the φ-space by the points $\varphi(y)$ nearest to the ideal point ω. Hence the whole problem consists of two stages:

(1) extremization (viz, maximization) of φ over X, i.e., selection of the set $V = \operatorname{Arg\,max}_{x \in X} \varphi(x)$, and

(2) extremization (viz, minimization) of the distance d from the points of the set $\varphi(V)$ to ω, i.e., maximization over V of the function $\psi(v) = -d(\varphi(v), \omega)$, or speaking differently, selection of the set $Y = \operatorname{Arg\,min}_{v \in V} d(\varphi(v), \omega) = \operatorname{Arg\,max}_{v \in V} \psi(v)$.

Consider the two cases.

Case (a) The ideal point ω is *not exceeded on the set* U:

$$\forall i\colon \omega_i \geqslant \varphi_i^*,$$

where

$$\varphi_i^* = \max_{x \in U} \varphi_i(x).$$

It is easy to see in this case that condition (56) of φ, ψ-consistency (absence of φ, ψ-inconsistent dyads) is satisfied. Hence due to Theorem 3 the given two-stage problem is reducible to its second stage, i.e., to finding elements x in X minimizing the distance $d(\varphi(x), \omega)$ from the point $\varphi(x)$ to the ideal point ω. Preliminary selection of the Pareto set by the vectorial criterion φ in this case is redundant from the theoretical point of view.

Case (b). The ideal point ω is *exceeded on* U at least in one coordinate i:

$$\exists i: \omega_i < \varphi_i^*.$$

In this case some φ, ψ-inconsistent dyad, or moreover, φ, ψ-1-inconsistent triad can be formed in U. An example of such a situation is presented in Fig. 1. In this particular example the function $\varphi(x)$ is two-dimensional, the scalar function $\psi(x)$ is defined as the Euclidean distance, with the opposite sign, on the plane between points $\varphi(x)$ and ω. For the configuration of points in the φ-space in Fig. 1, we have

$$d(\varphi(x), \omega) < d(\varphi(y), \omega) < d(\varphi(z), \omega),$$

and hence

$$\psi(x) > \psi(y) > \psi(z),$$

which together with vectorial relations

$$\varphi(x) >< \varphi(y), \qquad \varphi(y) >< \varphi(z), \qquad \varphi(x) < \varphi(z)$$

makes the triple x, y, z a φ, ψ-1-triad (with $u = x$, $v = z$, $w = y$ in definitions (8), (18.1)). Therefore, in virtue of Theorem 2 such a two-stage φ, ψ-problem not only is irreducible to its second stage, but moreover, it is not reducible to any one-stage extremization problem at all.

We can demonstrate this directly. It is evident that Pareto-optimal elements in the set $X = \{x, y, z\}$ are y and z but not x. In the pair y and z in φ-space, the nearest to the ideal ω turns out to be the element y which will be the only eventually chosen element from X. At the same time in a narrower set $X' = \{x, y\} \subset X$ both elements x, y will be Pareto-optimal, and the nearest to ω of these two will be not y but x which will be chosen

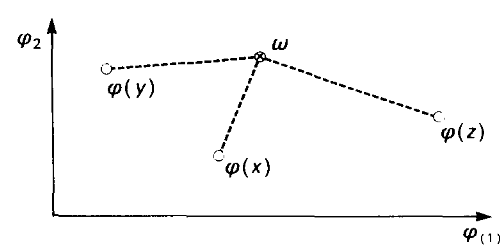

FIGURE 1

finally in spite of the presence of y in X'. If the given two-stage problem were reducible universally to some one-stage problem of extremization of some, even vectorial, function f, then the above solutions of the initial problem would have yielded the following. Due to the choice of y from $X=\{x, y, z\}$, we would have $f(y)>f(x)$, but due to the choice of x from $X'=\{x, y\}$, $f(x)>f(y)$. This is impossible if f does not depend on X.

So the consideration of both cases, (a) and (b), shows us that a two-stage problem of the "selection of the nearest to the ideal" points in the Pareto set can demonstrate different properties. Namely, depending on mutual disposition of the initial set and the ideal point, the problem can reveal both universal reducibility, and irreducibility to one-stage extremization.

Remark. In the considered example of a two-stage problem the ideal point ω is assumed to be fixed, independent of X. Sometimes one considers a little different statement of such a problem when ω is a function of X, which may be implicit. So, e.g., if we let

$$\forall i\colon \omega_i = \max_{x \in X} \varphi_i(x),$$

then $\omega=(\omega_1,..., \omega_n)$ obviously depends on X: $\omega=\omega(X)$. In this case the initial two-stage problem is not a φ, ψ-problem of the form (3). Indeed, its second stage, maximization of the function $\psi = -d(\varphi(x), \omega(X))$, is not an extremization problem of the form (1), because here the criterial function ψ does depend on X as on a parameter: $\psi=\psi_X(x)$. Hence a problem with an unfixed ideal point does not fall under the above scheme of analysis at all.

This remark once again reminds us that the abstract form of the extremization problem (1) adopted here, though very general, nevertheless imposes an essential requirement. This requirement lies in the criterial function independence of set X being variated in the "mass" statement of the problem. This requirement, pointed out in the Introduction (see Remark in Sect. (1), is just one in which underlies the exact statement of the question concerning the universal reducibility of a problem to one-stage extremization and makes the question nontrivially solvable in principle.

References

1. M. A. Aizerman and A. V. Malishevski, General theory of best variants choice: Some aspects, *IEEE Trans. Automat. Control*, **AC-26**, No. 5 (1981), 1030–1040.
2. M. A. Aizerman, "New Problems in the General Choice Theory," California Inst. of Tech., Social Sci. Work. Paper 560, 1985.
3. P. C. Fishburn, "Utility Theory for Decision Making," Wiley, New York, 1978.
4. A. K. Sen, "Collective Choice and Social Welfare," Holden Day, San Francisco, 1970.
5. O. Ore, "Theory of Graphs," Amer. Math Society, Providence, R.I., 1962.